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EQUILIBRIUM OF A NONHOMOGENEOUS HALF-PLANE UNDER THE ACTION OF FORCES APPLIED TO THE BOUNDARY

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By the method of Fourier integral transforms we construct the exact solution of the problem of equilibrium of a nonhomogeneous half-plane $z \geqslant 0$ under the action of normal and tangential forces applied to the boundary. The shear modulus of the half-plane is a power function of a linear binomial in the Cartesian coordinate z while Poisson ratio is constant.

In the papers [1 – 4], devoted to similar problems, the equilibrium of a half-plane and a half-space $z \geqslant 0$ with modulus of elasticity $E(z) = E_0 z^\kappa$, was investigated. It is obvious that such media are physically not real, since the modulus of elasticity is equal to zero on the surface. This circumstance, in particular, implies a restriction on the possible values of the exponent k. Thus, for example, the formulation of the problem on the action of a distributed load has sense only for $0 \leqslant k < 1$, which in turn, restricts considerably the sphere of applicability of the power law adapted by the authors as an interpolation formula which models the foundation. Therefore, it is of interest to investigate the equilibrium of a nonhomogeneous half-plane, whose modulus of elasticity varies with depth according to a power relation and it is different from zero on the surface. In [5] we have considered a similar problem for the half-space, but some restrictions were imposed on the Poisson's ratio.

1. We determine the state of stress and strain of a nonhomogeneous half-plane $z \geqslant 0$ under the action of forces applied to the boundary $z = 0$. The shear modulus of the half-plane varies according to the rule

$$G(z) = G_0 (1 + cz)^b \tag{1.1}$$

and the Poisson's ratio ν is constant.

The general solution of the plane problem of the theory of elasticity for the nonhomogeneous isotropic media, whose elastic characterisitics are differentiable functions of the Cartesian coordinate z, is of the form [6]

$$u_x = -\frac{1}{2G}\left[\nu^* \frac{\partial^2}{\partial x^2} - (1-\nu^*)\frac{\partial^2}{\partial z^2}\right]\frac{\partial L}{\partial x} \tag{1.2}$$

$$u_z = -\frac{1}{G}\frac{\partial^3 L}{\partial x^2 \partial z} + \frac{\partial}{\partial z}\left\{\frac{1}{2G}\left[\nu^* \frac{\partial^2 L}{\partial x^2} - (1-\nu^*)\frac{\partial^2 L}{\partial z^2}\right]\right\}$$

$$\sigma_x = \frac{\partial^4 L}{\partial x^2 \partial z^2}, \qquad \sigma_z = \frac{\partial^4 L}{\partial x^4}, \qquad \tau_{xz} = -\frac{\partial^4 L}{\partial x^3 \partial z}$$

Here u_x, u_z, σ_x, σ_z, τ_{xz} are the components of the displacement vector and of the stress tensor, $\nu^* = \nu$ for the case of plane strain, taking place in a plane parallel to xOz, and $\nu^* = \nu / (1 + \nu)$ for the case of the generalized plane state of stress, L is a function satisfying the equation

$$\Delta\left(\frac{1-\nu^*}{G}\Delta L\right) - \frac{\partial^2 L}{\partial x^2}\frac{d^2}{dz^2}\left(\frac{1}{G}\right) = 0 \qquad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) \tag{1.3}$$

Thus, the problem reduces to finding in the domain $z \geqslant 0$ such a solution of Eq. (1.3) which is bounded at infinity and which satisfies the following conditions at the boundary $z = 0$ of the half-plane:

$$\sigma_z|_{z=0} = \sigma(x), \qquad \tau_{xz}|_{z=0} = \tau(x) \tag{1.4}$$

where $\sigma(x)$ and $\tau(x)$ are the normal and tangential forces applied to the half-plane. We assume that the functions $\sigma(x)$ and $\tau(x)$ can be represented by Fourier integrals

$$\sigma(x) = \int_{-\infty}^{+\infty} g_1(\alpha) e^{-i\alpha x} d\alpha, \qquad \tau(x) = \int_{-\infty}^{+\infty} g_2(\alpha) e^{-i\alpha x} d\alpha \tag{1.5}$$

where

$$g_1(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \sigma(x) e^{i\alpha x} dx, \qquad g_2(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tau(x) e^{i\alpha x} dx$$

We seek the solution of Eq. (1.3) in the form

$$L = \int_{-\infty}^{+\infty} e^{-i\alpha x}\psi(z, \alpha)\frac{d\alpha}{\alpha^4} \tag{1.6}$$

Here $\psi(z, \alpha)$ is a function, bounded for $z \to \infty$ and which is subject to determination. Substituting the expressions (1.6) and (1.1) into Eq. (1.3) and the equalities (1.4), we find that the function $\psi(z, \alpha)$ is the sulution of the equation

$$\frac{d^4\psi}{dz^4} - \frac{2cb}{1+cz}\frac{d^3\psi}{dz^3} + \left[\frac{c^2b(b+1)}{(1+cz)^2} - 2\alpha^2\right]\frac{d^2\psi}{dz^2} + \frac{2\alpha^2cb}{1+cz}\frac{d\psi}{dz} + \alpha^2\left[\alpha^2 + \frac{\nu^*c^2b(b+1)}{(1-\nu^*)(1+cz)^2}\right]\psi = 0 \qquad (1.7)$$

with the following boundary conditions:

$$\psi|_{z=0} = g_1(\alpha), \qquad \left.\frac{d\psi}{dz}\right|_{z=0} = i\alpha g_2(\alpha) \qquad (1.8)$$

Thus, the plane problem of the theory of elasticity has been reduced to a boundary value problem for the ordinary differential equation (1.7). The latter, by the substitution $\eta = (1 + cz)\alpha / c$ reduces to the type investigated in [3, 6]. The solution, bounded for $z \to \infty$, has the form:

for $\nu^* \neq 1/(b+1)$ and $b \neq -1$

$$\psi(z, \alpha) = z_1^{\mu-1/2}[C_1W_{\chi,\mu}(2\eta) + C_2W_{-\chi,\mu}(2\eta)] \qquad (1.9)$$

for $\nu^* = 1/(b+1)$ or $b = -1$

$$\psi(z, \alpha) = z_1^{\mu}\left\{C_1K_\mu(\eta) + C_2\left[I_\mu(\eta)\int K_\mu^2(\eta)\,d\eta - K_\mu(\eta)\int I_\mu(\eta)K_\mu(\eta)\,d\eta\right]\right\} \qquad (1.10)$$

Here C_1, C_2 are unknown functions of the parameter α, which have to be determined from the boundary conditions, $I_\mu(\eta)$, $K_\mu(\eta)$ are Bessel functions of imaginary arguments of the first and second kind and of order μ, $W_{\pm\chi,\mu}(2\eta)$ are the Whittaker functions

$$\mu = 1 + b/2, \quad z_1 = 1 + cz, \quad \chi = {}^1/_2\sqrt{(b+1)[1 - \nu^*b/(1-\nu^*)]}$$

Let us assume for definiteness, that $\nu^* \neq 1/(b+1)$ and $b \neq -1$. Proceeding from the expressions (1.9) and (1.6), we take the function $\psi(z, \alpha)$ in the form

$$\psi(z, \alpha) = z_1^{\mu-1/2}[C_1W_{\chi,\mu}(2\lambda z_1) + C_2W_{-\chi,\mu}(2\lambda z_1)] \quad \left(\lambda = \frac{|\alpha|}{c}\right) \qquad (1.11)$$

For simplicity, we shall omit in the following the arguments $2\lambda z_1$ of the Whittaker function. Substituting $\psi(z, \alpha)$ into the conditions (1.8), we obtain a system of algebraic equations for the functions C_1 and C_2, from which we find

$$C_1 = \frac{g_1(\alpha)}{\Delta_1}\left[\left(\lambda + \mu + \chi - \frac{1}{2}\right)W_{-\chi,\mu}(2\lambda) - W_{-\chi+1,\mu}(2\lambda)\right] - \frac{i\alpha}{c\Delta_1}g_2(\alpha)W_{-\chi,\mu}(2\lambda) \qquad (1.12)$$

$$C_2 = -\frac{g_1(\alpha)}{\Delta_1}\left[\left(\lambda + \mu - \chi - \frac{1}{2}\right)W_{\chi,\mu}(2\lambda) - W_{\chi+1,\mu}(2\lambda)\right] + \frac{i\alpha}{c\Delta_1}g_2(\alpha)W_{\chi,\mu}(2\lambda)$$

$$\Delta_1 = W_{\chi,\mu}(2\lambda)[2\chi W_{-\chi,\mu}(2\lambda) - W_{-\chi+1,\mu}(2\lambda)] + W_{-\chi,\mu}(2\lambda)W_{\chi+1,\mu}(2\lambda)$$

Substituting the expression (1.11) into (1.6), we obtain the function L, from where, making use of the relations (1.2), we find the components of the displacement vector and the stress tensor

$$u_x = -\frac{iz_1^{-\mu-1/2}}{2G_0}\int_{-\infty}^{+\infty}\frac{e^{-i\alpha x}}{\alpha\lambda^2}\Big\{C_1\Big[(\lambda^2 z_1^2 + f_1^-)W_{\chi,\mu} - 2(1-\nu^*)\Big(\mu-\frac{1}{2}\Big)W_{\chi+1,\mu}\Big] + C_2\Big[(\lambda^2 z_1^2 + f_1^+)W_{-\chi,\mu} - 2(1-\nu^*)\Big(\mu-\frac{1}{2}\Big)W_{-\chi+1,\mu}\Big]\Big\}d\alpha \qquad (1.13)$$

$$u_z = \frac{cz_1^{-\mu-1/2}}{2G_0}\int_{-\infty}^{+\infty}\frac{e^{-i\alpha x}}{\alpha^2\lambda}\{C_1[(\lambda^2 z_1^2 + f_2^-)W_{\chi,\mu} - (\lambda z_1 + 2(1-\nu^*)\chi)W_{\chi+1,\mu}] + C_2[(\lambda^2 z_1^2 + f_2^+)W_{-\chi,\mu} - (\lambda z_1 - 2(1-\nu^*)\chi)W_{-\chi+1,\mu}]\}d\alpha$$

$$\sigma_x = -z_1^{\mu-5/2}\int_{-\infty}^{+\infty}\frac{e^{-i\alpha x}}{\lambda^2}\Big\{C_1\Big[\Big(\lambda^2 z_1^2 + \frac{f_1^-}{1-\nu^*}\Big)W_{\chi,\mu} - 2\Big(\mu-\frac{1}{2}\Big)W_{\chi+1,\mu}\Big] + C_2\Big[\Big(\lambda^2 z_1^2 + \frac{f_1^+}{1-\nu^*}\Big)W_{-\chi,\mu} - 2\Big(\mu-\frac{1}{2}\Big)W_{-\chi+1,\mu}\Big]\Big\}d\alpha$$

$$\sigma_z = z_1^{\mu-1/2}\int_{-\infty}^{+\infty}e^{-i\alpha x}[C_1 W_{\chi,\mu} + C_2 W_{-\chi,\mu}]\,d\alpha$$

$$\tau_{zx} = -icz_1^{\mu-3/2}\int_{-\infty}^{+\infty}\frac{e^{-i\alpha x}}{\alpha}\Big\{C_1\Big[\Big(\lambda z_1 + \mu - \chi - \frac{1}{2}\Big)W_{\chi,\mu} - W_{\chi+1,\mu}\Big] + C_2\Big[\Big(\lambda z_1 + \mu + \chi - \frac{1}{2}\Big)W_{-\chi,\mu} - W_{-\chi+1,\mu}\Big]\Big\}d\alpha$$

Here we have introduced the following notation:

$$f_1^{\pm} = 2(1-\nu^*)(\mu \pm \chi - 1/2)(\mu - 1/2 + \lambda z_1)$$

$$f_2^{\pm} = [(1+2\nu^*)(\mu \pm \chi - 1/2) \mp 2\chi - 1]\lambda z_1 \mp 2(1-\nu^*)\chi(\mu \pm \chi - 1/2)$$

where the signs in the left-hand and the right-hand sides correspond.

2. The asymptotic expansions of the Whittaker functions have the form [7]

$$W_{\chi,\mu}(2\lambda z_1) \sim (2\lambda z_1)^{\chi} e^{-\lambda z_1}\Big\{1 + \frac{\mu^2 - (\chi - 1/2)^2}{1!\,2\lambda z_1} + \frac{[\mu^2 - (\chi - 1/2)^2][\mu^2 - (\chi - 3/2)^2]}{2!\,(2\lambda z_1)^2} + \ldots\Big\} \qquad (2.1)$$

$$W_{-\chi,\mu}(2\lambda z_1) \sim (2\lambda z_1)^{-\chi} e^{-\lambda z_1}\Big\{1 + \frac{\mu^2 - (\chi + 1/2)^2}{1!\,2\lambda z_1} + \frac{[\mu^2 - (\chi + 1/2)^2][\mu^2 - (\chi + 3/2)^2]}{2!\,(2\lambda z_1)^2} + \ldots\Big\}$$

In some particular cases of nonhomogeneity of the elastic medium, the series terminate and formulas (2.1) become exact. In addition, the expressions (1.13) for the determination of the displacements and stresses can be significantly simplified. Let us find such cases by setting

$$\mu^2 - \left(\chi - \frac{2n-1}{2}\right)^2 = 0, \qquad \mu^2 - \left(\chi + \frac{2m-1}{2}\right)^2 = 0 \tag{2.2}$$

i.e. the first series terminates at the term with index n, while the second one at the term with index m. Hence, taking into account that $0 \leqslant \nu \leqslant {}^1/_2$ and $\chi > 0$ (for $\chi = 0$ either $\nu^* = 1/(b+1)$, or $b = -1$, and the function $\psi(z, \alpha)$ has to be taken in the form (1.10)), we find

$$b = m + n - 3, \qquad \nu^* = \frac{m + n - 2 - (n-m)^2}{4(m-1)(n-1)} \tag{2.3}$$

and for m and n we obtain a system of inequalities

$$n > m, \quad n \leqslant m + {}^1/_2[1 + \sqrt{8m-7}], \quad m \geqslant 2 \tag{2.4}$$

In addition, there exists one more solution of Eqs. (2.2) for the homogeneous medium

$$m = 1, \ n = 2, \ b = 0, \ 0 \leqslant \nu \leqslant {}^1/_2 \tag{2.5}$$

Since the constants G_0 and c, which characterize the elastic properties of the half-plane, do not depend on m and n there are no restrictions imposed on their magnitude. Below, for all possible $0 < b \leqslant 10$ we give the corresponding values of the generalized Posson's ratio ν^*, calculated with the formula (2.3)

2	3	4	5	6	7	8	8	9	10	10
$^1/_4$	0	$^1/_6$	$^1/_{18}$	$^1/_8$	$^1/_{15}$	0	$^1/_{10}$	$^1/_{16}$	$^1/_{56}$	$^1/_{12}$

Thus, for particular cases of nonhomogeneity of the elastic medium, defined by the relations (2.3) and (2.4), the Whittaker functions can be expressed in terms of exponential and power functions. Moreover, the integrands in the formulas (1.13) become simpler so that they can be integrated into special functions for many forms of the loads $\sigma(x)$ and $\tau(x)$.

We consider, for example, the case $b = 2$. Then $\nu^* = {}^1/_4$. Making use of the formulas (2,1), (1.12) and (1,13), we obtain the components of the displacement vector

$$u_x = -\frac{1}{2c(1+\zeta)G_0}\int_{-\infty}^{+\infty}\frac{e^{-i\alpha x}e^{-\lambda\zeta}}{2\lambda^2+6\lambda+3}\left\{\frac{i\alpha}{c}g_1(\alpha)[(2\lambda+1)\zeta-1] - g_2(\alpha)[2\lambda^2\zeta+3\lambda(\zeta-1)-6]\right\}d\alpha \tag{2.6}$$

$$u_z = -\frac{1}{2c(1+\zeta)G_0}\int_{-\infty}^{+\infty}\frac{e^{-i\alpha x}e^{-\lambda\zeta}}{2\lambda^2+6\lambda+3}\left\{g_1(\alpha)[2\lambda^2\zeta+\lambda(\zeta+3)+2] + \frac{i\alpha}{c}g_2(\alpha)[(2\lambda+3)\zeta+1]\right\}d\alpha \qquad (\zeta = cz)$$

For definiteness, we assume that concentrated loads are applied to the boundary of the half-plane. In further analysis it is convenient to consider separately the action of normal and tangential forces.

3. Assume that a concentrated force P is applied at the origin, normal to the half-plane and acting in the direction of the z-axis. Then

$$\sigma(x) = -P\delta(x), \qquad \tau(x) = 0 \tag{3.1}$$

where $\delta(x)$ is the Dirac delta function. Substituting the expressions (3.1) into (1.5), we obtain

$$g_1(\alpha) = -P/(2\pi), \quad g_2(\alpha) = 0 \tag{3.2}$$

Now, formulas (2.6) obtain the form

$$u_x = \frac{P}{2\pi(1+\zeta)G_0}\int_0^\infty \lambda \sin\lambda\xi \, e^{-\lambda\zeta}\frac{(2\lambda+1)\zeta-1}{2\lambda^2+6\lambda+3}\,d\lambda \tag{3.3}$$

$$u_z = \frac{P}{2\pi(1+\zeta)G_0}\int_0^\infty \cos\lambda\xi \, e^{-\lambda\zeta}\frac{2\lambda^2\zeta+\lambda(\zeta+3)+2}{2\lambda^2+6\lambda+3}\,d\lambda \qquad (\xi = cx)$$

We expand the rational functions in the integrands into simple fractions. Making use of the formulas [8]

$$\int_0^\infty \frac{e^{-\lambda p}}{\lambda+k}\,d\lambda = -e^{kp}\,\mathrm{Ei}(-kp) \qquad (|\arg k| < \pi, \ \mathrm{Re}\, p > 0) \tag{3.4}$$

$$\int_0^\infty \sin\lambda\xi\, e^{-\lambda\zeta}\,d\lambda = \frac{\xi}{\xi^2+\zeta^2}, \qquad \int_0^\infty \cos\lambda\xi\, e^{-\lambda\zeta}\,d\lambda = \frac{\zeta}{\xi^2+\zeta^2}$$

where $\mathrm{Ei}(-kp)$ is the exponential integral function, we obtain

$$u_z = \frac{P}{2\pi(1+\zeta)G_0}\left\{\frac{\zeta^2}{\xi^2+\zeta^2} + \frac{1}{12}\left[15\zeta - 9 + (9\zeta-5)\sqrt{3}\right] \times \right. \tag{3.5}$$

$$\left. E_1(\gamma_1\xi, \gamma_1\zeta) + \frac{1}{12}\left[15\zeta - 9 - (9\zeta-5)\sqrt{3}\right]E_1(\gamma_2\xi, \gamma_2\zeta)\right\}$$

$$u_x = \frac{P}{2\pi(1+\zeta)G_0}\left\{\frac{\xi\zeta}{\xi^2+\zeta^2} + \frac{1}{4}\left[5\zeta + 1 + (3\zeta+1)\sqrt{3}\right] \times \right.$$

$$\left. E_2(\gamma_1\xi, \gamma_1\zeta) + \frac{1}{4}\left[5\zeta + 1 - (3\zeta+1)\sqrt{3}\right]E_2(\gamma_2\xi, \gamma_2\zeta)\right\}$$

Here

$$E_1(\xi, \zeta) = \frac{1}{2}\left[e^u\,\mathrm{Ei}(-u) + e^{\bar u}\,\mathrm{Ei}(-\bar u)\right] \tag{3.6}$$

$$E_2(\xi, \zeta) = \frac{i}{2}\left[e^u\,\mathrm{Ei}(-u) - e^{\bar u}\,\mathrm{Ei}(-\bar u)\right]$$

$$u = \zeta + i\xi, \quad \bar u = \zeta - i\xi, \quad \gamma_1 = {}^1/_2\,(3+\sqrt{3}), \quad \gamma_2 = {}^1/_2\,(3-\sqrt{3})$$

We note that also for the other two particular cases of nonhomogeneity of the elastic medium, defined by the relations (2.3), the components of the displacement vector and stress tensor can be expressed in terms of the functions $E_1(\xi, \zeta)$ and $E_2(\xi, \zeta)$.

We introduce the polar system of coordinates

$$\rho = \sqrt{\xi^2+\zeta^2}, \qquad \varphi = \mathrm{arctg}\; x/z$$

If we make use now of the representation of the exponential integral function in the form of series, then from the expressions (3.6) we obtain ($\gamma = 1.781072418$ is Euler's constant)

$$E_1(\xi, \zeta) = e^{\zeta}\left[\ln\gamma\rho\cos\xi - \varphi\sin\xi + \sum_{k=1}^{\infty}(-\rho)^k\frac{\cos(k\varphi+\xi)}{k!\,k}\right] \tag{3.7}$$

$$E_2(\xi, \zeta) = -e^{\zeta}\left[\ln\gamma\rho\sin\xi + \varphi\cos\xi + \sum_{k=1}^{\infty}(-\rho)^k\frac{\sin(k\varphi+\xi)}{k!\,k}\right]$$

From the asymptotic expansion of the exponential integral function we obtain the asymptotic expansions for $E_1(\xi, \zeta)$ and $E_2(\xi, \zeta)$ at large ρ

$$E_1(\xi, \zeta) \sim -\frac{\cos\varphi}{\rho} + \frac{1!\cos 2\varphi}{\rho^2} - \frac{2!\cos 3\varphi}{\rho^3} + \ldots \tag{3.8}$$

$$E_2(\xi, \zeta) \sim -\frac{\sin\varphi}{\rho} + \frac{1!\sin 2\varphi}{\rho^2} - \frac{2!\sin 3\varphi}{\rho^3} + \ldots$$

If we make use of the formula [8]

$$\mathrm{Ei}\,(\pm i\xi) = \mathrm{ci}\,(\xi) \pm i\,\mathrm{si}\,(\xi)$$

where $\mathrm{si}(\xi)$, $\mathrm{ci}(\xi)$ are the integral sine and cosine functions, then for $\zeta = 0$ or $\xi = 0$ the functions $E_1(\xi, \zeta)$ and $E_2(\xi, \zeta)$ can be expressed in terms of the tabulated functions

$$E_1(\xi, 0) = \cos|\xi|\,\mathrm{ci}\,|\xi| + \sin|\xi|\,\mathrm{si}\,|\xi| \tag{3.9}$$

$$E_2(\xi, 0) = \begin{cases} \cos\xi\,\mathrm{si}\,\xi - \sin\xi\,\mathrm{ci}\,\xi, & \xi > 0 \\ -\cos|\xi|\,\mathrm{si}\,|\xi| + \sin|\xi|\,\mathrm{ci}\,|\xi|, & \xi < 0 \end{cases}$$

$$E_1(0, \zeta) = e^{\zeta}\,\mathrm{Ei}\,(-\zeta), \qquad E_2(0, \zeta) = 0$$

From the formulas (3.5) and (3.7) it follows that for $\rho \ll 1$

$$u_x \approx \frac{P}{4\pi G_0}(\sin 2\varphi - \varphi), \qquad u_z \approx \frac{P}{2\pi G_0}\left(\cos^2\varphi - \frac{3}{2}\ln r - C\right) \tag{3.10}$$

$$r = \sqrt{x^2 + z^2}, \qquad C = \frac{9 + 5\sqrt{3}}{12}\ln(c\gamma\Upsilon_1) + \frac{9 - 5\sqrt{3}}{12}\ln(c\gamma\Upsilon_2)$$

Thus, near the point of application of the force, the displacements of the nonhomogeneous half-plane coincide, except for a constant, with the displacements of an identically loaded homogeneous half-plane having the same Poisson's ratio and with shear modulus equal to G_0.

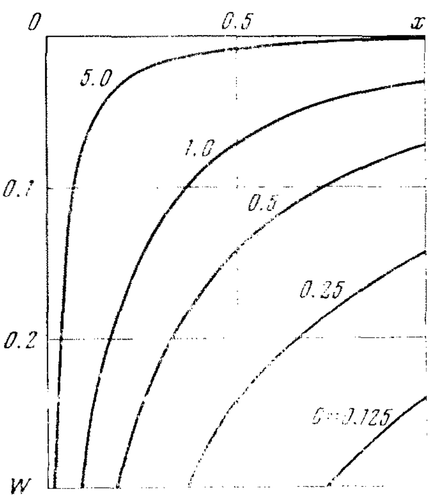

Fig. 1

As we move off the point of application of the force, the displacements decay fast. The character of the damping can be easily determined if we make use of the asymptotic expansions of the functions $E_1(\xi, \zeta)$ and $E_2(\xi, \zeta)$. Thus, for example, the displacements u_z at the boundary of the half-plane decrease for large ξ with a rate which is proportional to ξ^2.

Setting in the formulas (3.5) $c \to 0$, , we find that the displacements u_z increase indefinitely. However, if we fix one of the points of the half-plane, as this is done in solving of the corresponding problem for the homogeneous medium, we arrive at the well-known Flamant formulas.

The displacements of the boundary $z = 0$ of the half-plane, according to (3.9) and (3.5), are expressed in terms of tabulated functions and have the form

$$u_x\Big|_{z=0} = \frac{P}{8\pi G_0}\left[(1+\sqrt{3})\,E_2(\gamma_1\xi,\,0)+(1-\sqrt{3})\,E_2(\gamma_2\xi,\,0)\right] \tag{3.11}$$

$$u_z\Big|_{z=0} = -\frac{P}{24\pi G_0}\left[(9+5\sqrt{3})\,E_1(\gamma_1\xi,\,0)+(9-5\sqrt{3})\,E_1(\gamma_2\xi,\,0)\right]$$

In Fig. 1 we have represented the vertical displacements of the boundary of the half-plane for different values of the coefficient c. On the vertical axis we have represented the quantity $W = u_z G_0/P$.

4. Let us assume now that the concentrated force P, applied at the origin to the half-plane $z \geqslant 0$, acts in the positive direction of the x-axis, tangent to the boundary. Then

$$\sigma(x) = 0, \qquad \tau(x) = -P\delta(x) \tag{4.1}$$

Substituting the expressions (4.1) into (1.5), we obtain

$$g_1(\alpha) = 0, \qquad g_2(\alpha) = -P/(2\pi). \tag{4.2}$$

Now, formulas (2.6) can be transformed into

$$u_x = -\frac{P}{2\pi(1+\zeta)G_0}\int_0^\infty \cos\lambda\xi\, e^{-\lambda\zeta}\,\frac{2\lambda^2\zeta+3\lambda(\zeta-1)-6}{2\lambda^2+6\lambda+3}\,d\lambda \tag{4.3}$$

$$u_z = \frac{P}{2\pi(1+\zeta)G_0}\int_0^\infty \sin\lambda\xi\, e^{-\lambda\zeta}\lambda\,\frac{(2\lambda+3)\zeta+1}{2\lambda^2+6\lambda+3}\,d\lambda$$

We expand the rational functions in the integrands into simple fractions and we make use of the formulas (3.4). As a result we obtain

$$u_x = -\frac{P}{2\pi(1+\zeta)G_0}\left\{\frac{\zeta^2}{\xi^2+\zeta^2}+\frac{1}{4}\left[3(\zeta+1)+(\zeta-1)\sqrt{3}\right]\times E_1(\gamma_1\xi,\,\gamma_1\zeta)+\frac{1}{4}\left[3(\zeta+1)-(\zeta-1)\sqrt{3}\right]E_1(\gamma_2\xi,\,\gamma_2\zeta)\right\} \tag{4.4}$$

$$u_z = \frac{P}{2\pi(1+\zeta)G_0}\left\{\frac{\xi\zeta}{\xi^2+\zeta^2}+\frac{1}{4}\left[3\zeta-1+(\zeta-1)\sqrt{3}\right]\times E_2(\gamma_1\xi,\,\gamma_1\zeta)+\frac{1}{4}\left[3\zeta-1-(\zeta-1)\sqrt{3}\right]E_2(\gamma_2\xi,\,\gamma_2\zeta)\right\}$$

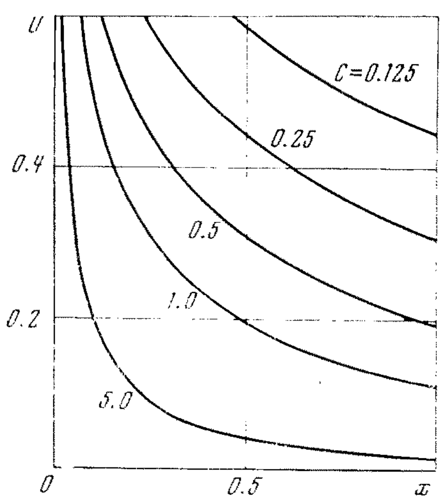

Fig. 2

The displacements of the boundary of the half-plane $z = 0$ can be expressed in terms of tabulated functions and have the form

$$u_x\Big|_{z=0} = -\frac{P}{4\pi G_0}\left[\gamma_2 E_1(\gamma_1\xi,0)+\gamma_1 E_1(\gamma_2\xi,0)\right] \tag{4.5}$$

$$u_z\Big|_{z=0} = -\frac{P}{8\pi G_0}\left[(1+\sqrt{3})\,E_2(\gamma_1\xi,\,0)+(1-\sqrt{3})\,E_2(\gamma_2\xi,\,0)\right]$$

From the expressions (4.4) and (3.7) it follows that near the point of application of the force $(\rho \ll 1)$ the displacements of the nonhomogeneous half-plane coincide, except for the constant, with the displacements of an identically loaded homogeneous half-plane, having the same Poisson's ratio and with shear modulus equal to G_0. However, as we move off the point of application of the force, the displacements start to decay fast. The character of the damping can be easily determined if we make use of the asymptotic expansions (3.8). Thus, for example, from the expressions (4.4) and (3.8) it follows that for $p \gg 1$ we have

$$u_x \big|_{z=0} \approx \frac{3P}{2\pi G_0 (cx)^2} \tag{4.6}$$

In Fig. 2 we have represented the displacements u_x of the points of the surface of the half-plane for different values of the coefficient c. On the vertical axis we have represented the quantity $U = u_x G_0 / P$.

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